ISSN 2181-3187

LOPITAL TEOREMASI VA UNING TADBIQLARI

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Annotasiya: Ushbu maqolada bir nechta aniqmaslikka ega bo'lgan funksiyalar limitini Lopital teoremasini qo'llagan holda hisoblash usullari keltirilgan.

Аннотация: В статье описаны методы вычисления предела нескольких неопределенностью функцей с помощью теоремы Лопитала

Annotation: The article describes methods for calculating the limit of some (complicated) function using the L'hopital's theorem

Kalit so'zlar: Differensiallanuvchi funksiya, hosila, aniqmaslikka ega bo'lgan funksiya , funksiya limiti,

Ключевые слова: Дифференциал для функции, производная, предел функции, вычисление пределов неопределенных форм

Keywords: Differential for function, derivative, limit of function, limits of indeterminate forms.

Lopital qoidasi

Oliy matematika sohasida matematik qoida ya'ni Lopital teoremasi (ya'ni Lopital qoidasi desak ham o'rinli bo'ladi; bundan keyin shunday ishlatamiz) aniqmaslikka ega bo'lgan ba'zi funksiyalarning limitini hosila yordamida hisoblash qoidasi bo'lib, ushbu teorema Giyom Lopital tomonidan 1696-yilda „Analyse des Infiniment Petits" darsligida nashr etilgan. ([1])

Funksiyalar limitlarini hisoblayotganda quyidagi ko'rinishdagi aniqmasliklar yuzaga kelishi mumkin: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 1^{∞}, 0^{0}, ∞^{0}. Ko'pincha $x\to a$ da $\frac{f(x)}{g(x)}$ nisbatning limitini topishga qaraganda $\frac{f'(x)}{g'(x)}$ nisbatning limitini topish oson bo'ladi. Bu nisbatlar limitlarinng tengligini **Lopital qoidalari** deb nomlanuvchi ushbu teoramalar ko'rsatadi. ([2])

1-Teorema. (a,b) intervalda aniqlangan, uzluksiz $f(x)$ va $g(x)$ funksiyalar uchun quyidagi shartlar bajarilgan bo'lsin:

1) $\lim\limits_{x\to a} f(x)=0,\ \lim\limits_{x\to a} g(x)=0;$

2) $\forall x\in(a,b)$ da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $g'(x)\neq 0$;

ISSN 2181-3187

3) $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}=k$ (k- chekli yoki cheksiz). U holda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=k$$

o‘rinli bo‘ladi.

Isbot. ◄ $f(x)$ va $g(x)$ funksiyalarning $x=a$ nuqtada qiymatlari nolga teng, ya'ni

$$f(a)=0,\quad g(a)=0 \tag{1}$$

deb olsak, natijada

$$\lim_{x\to a}f(x)=0=f(a),\quad \lim_{x\to a}g(x)=0=g(a)$$

tengliklar o‘rinli bo‘lib, $f(x)$ va $g(x)$ funksiyalar $[a,b)$ oraliqda uzluksiz bo‘ladi. $\forall x\in(a,b)$ nuqta olib, $[a,x]$ segmentda $f(x)$ va $g(x)$ funksiyalarni qaraymiz. Bu segmentda $f(x)$ va $g(x)$ funksiyalar Koshi teoremasining sartlarini qanoatlantiradi. Koshi teoremasiga ko‘ra a bilan x orasida shunday c $(a<c<x)$ nuqta topiladiki, ushbu

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(c)}{g'(c)}$$

.

tengliklar o‘rinli bo‘ladi. Bu tenglikdan esa (1) ga ko‘ra

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}$$

bo‘lishi kelib chiqadi. Ravshanki, $x\to a$ da $c\to a$. Demak,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{c\to a}\frac{f'(c)}{g'(c)}=k$$ ►

1-misol. Ushbu

$$\lim_{x\to\frac{\pi}{4}}\frac{\sqrt[3]{tgx}-1}{2\sin^2 x-1}$$

limitni hisoblang. ([3])

◄ $f(x)=\sqrt[3]{tgx}-1,\quad g(x)=2\sin^2 x-1$ funksiyalari uchun 1-teoremaning barcha shartlari bajariladi:

1) $\lim\limits_{x\to\frac{\pi}{4}}f(x)=\lim\limits_{x\to\frac{\pi}{4}}\left[\sqrt[3]{tgx}-1\right]=0,$

$\lim\limits_{x\to\frac{\pi}{4}}g(x)=\lim\limits_{x\to\frac{\pi}{4}}(2\sin^2 x-1)=0;$

2) $f'(x)=\frac{1}{3\sqrt[3]{tg^2x}}\sec^2 x,\quad g'(x)=4\sin x\cos x,\qquad g'(x)\neq 0;$

3) $\lim\limits_{x\to\frac{\pi}{4}}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to\frac{\pi}{4}}\frac{(\sqrt[3]{tgx}-1)'}{(2\sin^2 x-1)'}=\lim\limits_{x\to\frac{\pi}{4}}\frac{\frac{1}{3\sqrt[3]{tg^2x}}\sec^2 x}{4\sin x\cos x}=\frac{1}{3}$

Demak,

$$\lim_{x\to\frac{\pi}{4}}\frac{f'(x)}{g'(x)}=\lim_{x\to\frac{\pi}{4}}\frac{(\sqrt[3]{tgx}-1)'}{(2\sin^2 x-1)'}=\lim_{x\to\frac{\pi}{4}}\frac{\frac{1}{3\sqrt[3]{tg^2x}}\sec^2 x}{4\sin x\cos x}=\frac{1}{3}$$ ►

2-Teorema. $(c,+\infty)$ intervalda aniqlangan, uzluksiz $f(x)$ va $g(x)$ funksiyalar uchun quyidagi shartlar bajarilgan bo‘lsin:

1) $\lim\limits_{x\to\infty} f(x)=0,\ \lim\limits_{x\to\infty} g(x)=0;$

2) $\forall x\in(c,+\infty)$ da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $g'(x)\neq 0$;

3) $\lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}=k$ (k- chekli yoki cheksiz)

U holda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=k$$

o‘rinli bo‘ladi.

◄ $a>0$ deb, $t=\frac{1}{x}$ deymiz. Unda $t\in(0,\frac{1}{a})$ bo‘lib, $x\to+\infty$ da $t\to+0$. Endi $F(t)$ va $G(t)$ funksiyalarni quyidagicha

$$F(t)=f(\tfrac{1}{t}),\qquad G(t)=g(\tfrac{1}{t})$$

aniqlaymiz. Unda

$$t\to+0 \text{ da } F(t)\to 0,\quad G(t)\to 0;$$

$$F'(t)=f'(\tfrac{1}{t})\cdot(-\frac{1}{t^2}),\quad G'(t)=g'(\tfrac{1}{t})\cdot(-\frac{1}{t^2});$$

$$\frac{F'(t)}{G'(t)}=\frac{f'(\frac{1}{t})}{g'(\frac{1}{t})}\to k,\quad (t\to+0)$$

bo‘lib, 1-teoremaga ko‘ra, $t\to+0$ da

$$\frac{F(t)}{G(t)}\to k$$

bo‘ladi. Keyingi munosabatdan esa

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=k$$

bo‘lishi kelib chiqadi ►

ISSN 2181-3187

2-misol. Ushbu

$$\lim_{x\to+\infty}\frac{e^{\frac{1}{x^2}}-1}{2\operatorname{arctg}x^2-\pi}$$

limitni hisoblang. ([2])

◄Agar $f(x)=e^{\frac{1}{x^2}}-1$, $g(x)=2\operatorname{arctg}x^2-\pi$ deyilsa, ular uchun 2-teoremaning barcha shartlari bajariladi, jumladan

$$f'(x)=-\frac{2}{x^3}e^{\frac{1}{x^2}},\qquad g'(x)=\frac{4x}{1+x^4}$$

bo‘lib,

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{-\frac{2}{x^3}e^{\frac{1}{x^2}}}{\frac{4x}{1+x^4}}=-\lim_{x\to+\infty}\frac{1+x^4}{2x^4}=-\frac{1}{2}$$

bo‘ladi. 2-teoremaga ko‘ra

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{e^{\frac{1}{x^2}}-1}{2\operatorname{arctg}x^2-\pi}=-\frac{1}{2}$$

bo‘ladi. ►

Quyidagi teoremalar ham yuqorida keltirilgan teoremalarga o‘xshash isbotlanadi.

3-Teorema. (a,b) intervalda $f(x)$ va $g(x)$ funksiyalar uchun quyidagi shartlar bajarilgan bo‘lsin:

1) $\lim_{x\to a}f(x)=\infty,\ \lim_{x\to a}g(x)=\infty;$

2) (a,b) da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $g'(x)\neq 0$;

3) $\lim_{x\to a}\frac{f'(x)}{g'(x)}=k$ (k- chekli yoki cheksiz). U holda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=k$$

tenglik o‘rinli bo‘ladi.

4-Teorema. $(c,+\infty)$ intervalda chekli $f(x)$ va $g(x)$ funksiyalar uchun quyidagi shartlar bajarilgan bo‘lsin:

1) $\lim_{x\to+\infty}f(x)=\infty,\ \lim_{x\to+\infty}g(x)=\infty;$

2) $(c,+\infty)$ da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va $g'(x)\neq 0$;

3) $\lim\limits_{x\to+\infty}\frac{f'(x)}{g'(x)}=k$ (k- chekli yoki cheksiz)

U holda

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=k$$

o‘rinli bo‘ladi.

3-misol. Ushbu

$$\lim_{x\to+\infty}\frac{\ln x}{x^{\varepsilon}}\quad(\varepsilon>0)$$

$0\cdot\infty$

limitni hisoblang. ([3])

◀ $f(x)=\ln x,\quad g(x)=x^{\varepsilon}$ funksiyalari uchun 4-teoremaning barcha shartlari bajariladi:

1) $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}(\ln x)=\infty,$

$\lim\limits_{x\to+\infty}g(x)=\lim\limits_{x\to+\infty}(x^{\varepsilon})=\infty;$

2) $f'(x)=\frac{1}{x},\quad g'(x)=\varepsilon x^{\varepsilon-1},\qquad g'(x)\neq 0;$

3) $\lim\limits_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to+\infty}\frac{(\ln x)'}{(x^{\varepsilon})'}=\lim\limits_{x\to+\infty}\frac{\frac{1}{x}}{\varepsilon x^{\varepsilon-1}}=0$

Demak,

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{(\ln x)'}{(x^{\varepsilon})'}=\lim_{x\to+\infty}\frac{\frac{1}{x}}{\varepsilon x^{\varepsilon-1}}=0$$ ▶

2^0. $0\cdot\infty,\ \infty-\infty,\ 1^{\infty},\ 0^0$ **ko‘rinishidagi hollar.** Bu ko‘rinish-dagi aniqmasliklar $\frac{0}{0},\ \frac{\infty}{\infty}$ hollarga keltirilib, so‘ng yuqoridagi teoremalar qo‘llaniladi.

1) $\lim\limits_{x\to a}f(x)=0,\ \lim\limits_{x\to a}g(x)=\infty$ bo‘lganda $f(x)\cdot g(x)$ ifoda $0\cdot\infty$ ko‘rinishdagi aniqmaslik bo‘lib, funksiyaning limitini topish uchun uni

$$f(x)\cdot g(x)=\frac{f(x)}{\frac{1}{g(x)}}=\frac{g(x)}{\frac{1}{f(x)}}$$

deb, so‘ng 1- yoki 2-teoremalar qo‘llaniladi.

ISSN 2181-3187

Shuningdek, $\lim\limits_{x\to a} f(x)=+\infty,\ \lim\limits_{x\to a} g(x)=+\infty$ bo‘lganda $f(x)-g(x)$ $f(x)\cdot g(x)$ ifoda $\infty-\infty$ ko‘rinishdagi aniqmaslik bo‘lib funksiyaning limitini topish uchun uni

$$f(x)-g(x)=\frac{\dfrac{1}{g(x)}-\dfrac{1}{f(x)}}{\dfrac{1}{f(x)}\cdot\dfrac{1}{g(x)}}$$

deb, so‘ng 1-teorema qo‘llaniladi.

3) $x\to a$ da $f(x)\to 0,\ g(x)\to 0$ hamda $x\to a$ da $f(x)\to 1,\ g(x)\to +\infty$ bo‘lganda $\left(f(x)\right)^{g(x)}$ funksiyaning limitini topish uchun avvalo

$$y=(f(x))^{g(x)}$$

funksiya logarifmlanadi, so‘ng yuqoridagi teoremalar qo‘llaniladi.

3-misol. Ushbu

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$$

limit hisoblansin.

◀Avvalo $y=\lim\limits_{x\to 0}\left(\dfrac{\arcsin x}{x}\right)^{\frac{1}{x^2}}$ deb olamiz. Ravshanki, $x\to 0$ da

$$f(x)=\frac{\arcsin x}{x}\to 1,\qquad g(x)=\frac{1}{x^2}\to +\infty.$$

Sodda hisoblashlar yordamida topamiz:

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln\dfrac{\arcsin x}{x}}{x^2}=\lim_{x\to 0}\frac{\left(\ln\dfrac{\arcsin x}{x}\right)'}{\left(x^2\right)'}=$$

$$=\lim_{x\to 0}\frac{\arcsin x}{2(2-3x^2)\arcsin x+2x\sqrt{1-x^2}}=\lim_{x\to 0}\frac{\dfrac{1}{\sqrt{1-x^2}}}{-12x\arcsin+\dfrac{2(2-3x^2)}{\sqrt{1-x^2}}+2\sqrt{1-x^2}-\dfrac{2x^2}{\sqrt{1-x^2}}}=\frac{1}{6}$$

$$=\lim_{x\to 0}\frac{x-\sqrt{1-x^2}\arcsin x}{2x^2\sqrt{1-x^2}\arcsin x}=\lim_{x\to 0}\frac{1-1+\dfrac{x}{\sqrt{1-x^2}}\arcsin x}{(4x\sqrt{1-x^2}-\dfrac{2x^3}{\sqrt{1-x^2}})arcsinx+2x^2}=$$

$$=\lim_{x\to 0}\frac{\arcsin x}{2(2-3x^2)\arcsin x+2x\sqrt{1-x^2}}=\lim_{x\to 0}\frac{\dfrac{1}{\sqrt{1-x^2}}}{-12x\arcsin+\dfrac{2(2-3x^2)}{\sqrt{1-x^2}}+2\sqrt{1-x^2}-\dfrac{2x^2}{\sqrt{1-x^2}}}=\frac{1}{6}$$

Demak, $\lim_{x\to 0}\left(\frac{\arcsin x}{x}\right)^{\frac{1}{x^2}}=\frac{1}{6}$ ►

Foydalanilgan adabiyotlar.

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