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APPLICATIONS OF VECTORS IN PROOFS AND PROBLEM SOLVING

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Abstract: In the paper, we show how vectors can be used to prove theorem and in solving geometric problems.

Keywords: *vector, collinear vectors, dot product, scalar square, midline of a triangle, median*

The concept of a vector is one of the most important ones in mathematics. It arose in connection with the study of quantities that are characterized by numerical value and direction. This concept has developed due to its wide use in various fields not only of mathematics, but also of other sciences. It is undeniable the importance of vectors in physics, chemistry, economics, and biology.

We first encounter the concept of a vector in a school course in physics (when studying speed, force and acceleration). In the course of mathematics, this concept is used later. The study of vectors in higher education leads to the understanding that a vector is a purely mathematical concept that is used in physics or other applied sciences and which makes it possible to simplify the solution of complex problems not only in these sciences, but also in mathematics.

Let's give some well-known facts about the use of vectors [1].

Consider two lines , say *a* and *b*, and let a vector $\overrightarrow{AB}$ lie on the line *a*, and a vector $\overrightarrow{CD}$ lie on the line *b*.

- to prove that lines *a* and *b* perpendicular, it is sufficient to prove that the dot product of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is equal to zero;
- to prove that lines *a* and *b* parallel, it is sufficient to prove the collinearity of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$;

Now consider points *A*, *B*, *C*, and *D*.

- to prove that *A*, *B*, and *C* lie on the one line, it is sufficient to establish the validity of one of the following equalities $\overrightarrow{AB} = k \cdot \overrightarrow{BC}$, or $\overrightarrow{AC} = k \cdot \overrightarrow{BC}$, or $\overrightarrow{AC} = k \cdot \overrightarrow{AB}$;
- to prove that a plane can be drawn through points *A*, *B*, *C*, and *D* , it is sufficient to prove the coplanarity of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ or three other vectors emanating from one point with ends at three other points.

In the present paper, we give examples of the use of vectors in the proof and solution of geometric problems.

First, consider the proof of the following

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Theorem. *The midline of a triangle is parallel to its third side and equal to half of it.*

Proof. Consider an arbitrary triangle *ABC* (see Fig. 1) and introduce the following vectors $\overrightarrow{AB}=\vec{c}$, $\overrightarrow{BC}=\vec{a}$, and $\overrightarrow{AC}=\vec{b}$. Then by definition of the sum of vectors, we have $\vec{c}+\vec{a}=\vec{b}$.

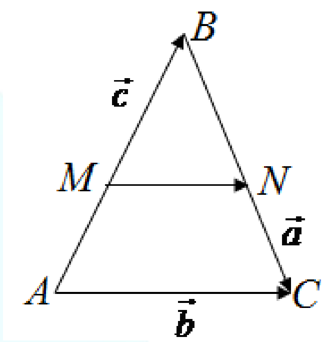

Figure 1.

Further, let *M* and *N* be the midpoints of the sides *AB* and *BC*, respectively. Then, obviously, $\overrightarrow{MN}=\overrightarrow{MB}+\overrightarrow{BN}$. By the definition of multiplying a vector by a number, $\overrightarrow{MB}=\frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BN}=\frac{1}{2}\overrightarrow{BC}$, therefore $\overrightarrow{MN}=\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}$ or

$$\overrightarrow{MN}=\frac{\vec{c}}{2}+\frac{\vec{a}}{2}=\frac{1}{2}(\vec{c}+\vec{a})=\frac{1}{2}\vec{b}. \qquad (1)$$

Taking into account that $\overrightarrow{AC}=\vec{b}$, we obtain from equality (1) that $\overrightarrow{MN}=\frac{1}{2}\overrightarrow{AC}$. Thus, we prove that the midline *MN* of the considered triangle *ABC* is equal to half the side *AC*.

Further, the collinearity of the vectors $\overrightarrow{AC}$ and $\overrightarrow{MN}$ follows from equality (1) and the definition of multiplication of a vector by a number, due to which the median line *MN* is parallel to the side *AC*. Theorem is proved.

The next application of vectors is the use of the concept of dot product of vectors in solving the following

Problem. Find the sum of the squares of the medians of a triangle if its sides *a*, *b* and *c* are known.

Solution. Consider an arbitrary triangle *ABC* (see Fig. 2) and introduce the following notations: $\overrightarrow{AB}=\vec{c}$, $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$. By definition of the sum of vectors, we have

$$\overrightarrow{AD}=\vec{c}+\frac{\vec{a}}{2},\ \overrightarrow{BE}=\vec{a}+\frac{\vec{b}}{2},\ \overrightarrow{CF}=\vec{b}+\frac{\vec{c}}{2}.$$

Using the properties of the scalar square, we obtain:

$$\overrightarrow{AD}^2+\overrightarrow{BE}^2+\overrightarrow{CF}^2=\left(\vec{c}+\frac{\vec{a}}{2}\right)^2+\left(\vec{a}+\frac{\vec{b}}{2}\right)^2+\left(\vec{b}+\frac{\vec{c}}{2}\right)^2=$$

$$\vec{c}^{\,2}+\vec{c}\cdot\vec{a}+\frac{\vec{a}^{\,2}}{4}+\vec{a}^{\,2}+\vec{a}\cdot\vec{b}+\frac{\vec{b}^{\,2}}{4}+\vec{b}^{\,2}+\vec{b}\cdot\vec{c}+\frac{\vec{c}^{\,2}}{4}=$$

$$\frac{5}{4}\left(a^2+b^2+c^2\right)+\left(\vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}\right),$$

i.e.

$$\overrightarrow{AD}^2+\overrightarrow{BE}^2+\overrightarrow{CF}^2=\frac{5}{4}\left(a^2+b^2+c^2\right)+\left(\vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}\right). \qquad \textbf{(2)}$$

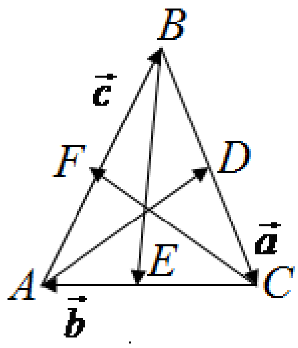

Figure 2.

In the last equality, we use the scalar square property: $\vec{a}^{\,2}=a^2$, where a is the length of the vector $\vec{a}$.

Since according to the vector addition rule,

$$\vec{a}+\vec{b}+\vec{c}=\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=0,$$

using the property of the dot product of vectors, we get

$$\left(\vec{a}+\vec{b}+\vec{c}\right)^2=0.$$

Then, according to the rule of squaring the sum of three terms, we obtain

$$\vec{a}^{\,2}+\vec{b}^{\,2}+\vec{c}^{\,2}+2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=0,$$

or

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{a^2+b^2+c^2}{2}.$$

Equality (2) implies that

$$|AD|^2+|BE|^2+|CF|^2=\frac{3}{4}\left(a^2+b^2+c^2\right).$$

Here we again use the property of the scalar square, according to which

$$\overrightarrow{AD}^2=|AD|^2,\quad \overrightarrow{BE}^2=|BE|^2,\quad \overrightarrow{CF}^2=|CF|^2.$$

Thus, using the properties of the dot product of vectors, we have found that the sum of the squares of the medians of an arbitrary triangle is equal to three fourths of the sum of the squares of its sides.

The considered examples clearly show that the use of vectors and the properties of various operations on them is a powerful tool for solving geometric problems, including proofs.

References

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