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THE METHOD OF MATHEMATICAL INDUCTION FOR DIOPHANTINE EQUATIONS

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Abstract: It is known that the mathematical induction method has several practical applications in solving problems and assertions related to non-negative integers. The main task of this article is the methods of solving the given equation in the case of additional non-negative integer parameters of Diophantine equations, the number of solutions and analyze under what conditions they do not have a solution.
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Introduction

Mathematical induction is a powerful and elegant method for proving statements depending on nonnegative integers.

Let $\big(P(n)\big)_{n\geq 0}$ be a sequence of propositions. The method of mathematical induction assists us in proving that $P(n)$ is true for all $n \geq n_0$, where n_0 is a given nonnegative integer.

Mathematical Induction (weak form): Suppose that:

• $P(n_0)$ is true;

• For all $k \geq n_0$, $P(k)$ is true implies $P(k+1)$ is true.

Then P(n) is true for all $n \geq n_0$.

Mathematical Induction: Let s be a fixed positive integer. Suppose that:

• $P(n_0)$, $P(n_0+1)$,..., $P(n_0+s-1)$ are true;

• For all $k \geq n_0$, $P(k)$ is true implies $P(k+s)$ is true.

Then P(n) is true for all $n \geq n_0$.

Mathematical Induction (strong form): Suppose that

• $P(n_0)$ is true;

• For all $k \geq n_0$, $P(m)$ is true for all m with $n_0 \leq m \leq k$ implies $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.

This method of proof is widely used in various areas of mathematics, including number theory. The following examples are meant to show how mathematical induction works in studying Diophantine equations.

Example 1. Prove that for all integers $n \geq 3$, there exist odd positive integer x, y, such that $7x^2 + y^2 = n^2$.

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(Bulgarian Mathematical Olympiad)

Solution. We will prove that there exist odd positive integers x_n, y_n such that

$7x_n^2 + y_n^2 = 2^n$.

For $n = 3$, we have $x_3 = y_3 = 1$. Now suppose that for a given integer $n \geq 3$ we have odd integers x_n, y_n satisfying $7x^2 + y^2 = n^2$. We shall exhibit a pair (x_{n+1}, y_{n+1}) of odd positive integers such that $7x_{n+1}^2 + y_{n+1}^2 = 2^{n+1}$. In fact,

$$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n + y_n}{2}\right)^2 = 2(7x_n^2 + y_n^2) = 2^{n+1}.$$

Precisely one of the numbers $\frac{x_n+y_n}{2}$ and $\frac{|x_n-y_n|}{2}$ is odd (since their sum is the larger of x_n and y_n, which is odd). If, for example, $\frac{x_n+y_n}{2}$ is odd, then

$$\frac{7x_n - y_n}{2} = 3x_n + \frac{x_n - y_n}{2}$$

is also odd (as a sum of an odd and an even number); hence in this case we may choose

$x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = \frac{7x_n-y_n}{2}$.

If $\frac{x_n-y_n}{2}$ is odd, then

$$\frac{7x_n + y_n}{2} = 3x_n + \frac{x_n + y_n}{2}$$

so we can choose

$x_{n+1} = \frac{|x_n-y_n|}{2}$ and $y_{n+1} = \frac{7x_n+y_n}{2}$

Example 2. Prove that for all positive integers n, the equation

$$x^2 + y^2 + z^2 = 59^n$$

is solvable in positive integers.

Solution. We use mathematical induction with pace $s = 2$ and $n_0 = 1$. Note that for $(x_1, y_1, z_1) = (1,3,7)$ and $(x_2, y_2, z_2) = (14,39,42)$ we have

$x_1^2 + y_1^2 + z_1^2 = 59$ and $x_2^2 + y_2^2 + z_2^2 = 59^2$.

Define now $(x_n, y_n, z_n), n \geq 3$, by

$x_{n+2} = 59x_n$, $y_{n+2} = 59y_n$, $z_{n+2} = 59z_n$,

for all $n \geq 1$. Then

$$x_{k+2}^2 + y_{k+2}^2 + z_{k+2}^2 = 59^2(x_k^2 + y_k^2 + z_k^2);$$

hence $x_k^2 + y_k^2 + z_k^2 = 59^k$ implies $x_{k+2}^2 + y_{k+2}^2 + z_{k+2}^2 = 59^{k+2}$.

Remark. We can write the solutions as

$$(x_{2n-1}, y_{2n-1}, z_{2n-1}) = (1 \cdot 59^{n-1}, 3 \cdot 59^{n-1}, 7 \cdot 59^{n-1})$$

and

$(x_{2n}, y_{2n}, z_{2n}) = (14 \cdot 59^n, 39 \cdot 59^n, 42 \cdot 59^n)$, $n \geq 1$.

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Example 3. Prove that for all $n \geq 3$ the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1 \qquad (1)$$

is solvable in distinct positive integers.

Solution. For the base case $n=3$ we have

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1.$$

Assuming that for some $k \geq 3$,

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}=1,$$

where $x_1, x_2, \ldots, x_k$ are distinct positive integers, we obtain

$$\frac{1}{2x_1}+\frac{1}{2x_2}+\cdots+\frac{1}{2x_k}=\frac{1}{2}.$$

It follows that

$$\frac{1}{2}+\frac{1}{2x_1}+\frac{1}{2x_2}+\cdots+\frac{1}{2x_k}=1,$$

where $2, 2x_1, 2x_2, \ldots, 2x_k$ are distinct.

Remarks. (1) Note that

$$\sum_{k=1}^{n-1}\frac{k}{(k+1)!}=\sum_{k=1}^{n-1}\frac{(k+1)-1}{(k+1)!}=\sum_{k=1}^{n-1}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right)=1-\frac{1}{n!}.$$

Hence

$$\frac{1}{\frac{2!}{1}}+\frac{1}{\frac{3!}{2}}+\cdots+\frac{1}{\frac{n!}{n-1}}+\frac{1}{n!}=1$$

i.e., $\left(\frac{2!}{1},\frac{3!}{2},\ldots,\frac{n!}{n-1},n!\right)$ is a solution to equation (1) and all its components are distinct.

(2) Another solution to equation (1) whose components are distinct is given by

$$\left(2,2^2,\ldots,2^{n-2},2^{n-2}+1,2^{n-2}(2^{n-2}+1)\right).$$

Indeed,

$$\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-2}}+\frac{1}{2^{n-2}+1}+\frac{1}{2^{n-2}(2^{n-2}+1)}$$

$$=1-\frac{1}{2^{n-2}}+\frac{2^{n-2}}{2^{n-2}(2^{n-2}+1)}+\frac{1}{2^{n-2}(2^{n-2}+1)}$$

$$=1-\frac{1}{2^{n-2}}+\frac{1}{2^{n-2}}=1.$$

(3) Another way to construct solutions to equation (1) is to consider the sequence

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$$a_1=2,\quad a_{m+1}=a_1\cdots a_m+1,\quad m\geq 1.$$

Then for all $n\geq 3$,

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}+\frac{1}{a_n-1}=1. \qquad (2)$$

Indeed, from the recurrence relation it follows that

$$a_{k+1}-1=a_k(a_k-1),\qquad k\geq 1,$$

or

$$\frac{1}{a_{k+1}-1}=\frac{1}{a_k-1}-\frac{1}{a_k},\qquad k\geq 1.$$

Thus

$$\frac{1}{a_k}=\frac{1}{a_k-1}-\frac{1}{a_{k+1}-1}$$

and the sum

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}$$

telescopes to

$$\frac{1}{a_1-1}-\frac{1}{a_n-1}=1-\frac{1}{a_n-1}.$$

Hence the relation (2) is verified.

(4) If $(s_1,s_2,\ldots,s_n)$ is a solution to

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1$$

with $s_1<s_2<\cdots<s_n$, then $\big(s_1,s_2,\ldots,s_{n-1},s_n+1,s_n(s_n+1)\big)$ is a solution to

$$\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_{n+1}}=1$$

and all its components are distinct.

(5) For $a>1$, the identity

$$\frac{1}{a-1}=\frac{1}{a}+\frac{1}{a^2}+\cdots+\frac{1}{a^m}+\frac{1}{(a-1)a^m}$$

generates various other families of solutions. For example, from

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1.$$

and $a=7$, we obtain the solution $(2,3,7,7^2,\ldots,7^{n-3},6\cdot 7^{n-3})$, $n\geq 4$, while from

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=1$$

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we get $(2,3,7,43,43^2,\dots,43^{n-4},42\cdot 43^{n-4})$, $n\geq 5$. From the construction above it follows that equation (1) has infinitely many families of solutions with distinct components.

(6) It is not known whether there are infinitely many positive integers n for which equation (1) admits solutions $(x_1,x_2,\dots,x_n)$, where $x_1,x_2,\dots,x_n$ are all distinct odd positive integers.

A simple parity argument shows that in this case n must be odd.

There are several known examples of such integers n. For instance, if $n=9$, we have

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{33}+\frac{1}{45}+\frac{1}{385}=1;$$

if $n=11$,

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{15}+\frac{1}{21}+\frac{1}{27}+\frac{1}{35}+\frac{1}{63}+\frac{1}{105}+\frac{1}{135}=1;$$

if $n=15$,

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{15}+\frac{1}{21}+\frac{1}{35}+\frac{1}{45}+\frac{1}{55}$$
$$\frac{1}{77}+\frac{1}{165}+\frac{1}{231}+\frac{1}{385}+\frac{1}{495}+\frac{1}{693}=1;$$

and if $n=17$,

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{15}+\frac{1}{21}+\frac{1}{35}+\frac{1}{45}+\frac{1}{55}$$
$$\frac{1}{77}+\frac{1}{165}+\frac{1}{275}+\frac{1}{385}+\frac{1}{495}+\frac{1}{825}+\frac{1}{1925}+\frac{1}{2475}=1;$$

Example 4. Prove that equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\cdots+\frac{1}{x_n^2}=\frac{n+1}{x_{n+1}^2}$$

is solvable in positive integers if and only if $n\geq 3$.

(Mathematical Reflections)

Solution. For, $n=1$ the equation becomes

$$\frac{1}{x_1^2}=\frac{2}{x_2^2},$$

which has no solution, since $\sqrt{2}$ is irrational.

Consider next $n=2$. Then the equation becomes

$$(x_2x_3)^2+(x_1x_3)^2=3(x_1x_2)^2.$$

For $1\leq i\leq 3$, write $x_i=3^{n_i}y_i$, where y_i is not divisible by 3. Without loss of generality assume that $n_1\geq n_2$. Then

$$3^{2(n_2+n_3)}\left((y_2y_3)^2+3^{2(n_1-n_2)}(y_1y_3)^2\right)=3^{2(n_1+n_2)+1}(y_1y_2)^2. \qquad (3)$$

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Because 1 is the only possible quadratic residue modulo 3,

$$(y_2y_3)^2 + 3^{2(n_1-n_2)}(y_1y_3)^2 \equiv 1 \text{ or } 2 \pmod 3.$$

Hence the exponents of 3 in the two sides of (3) cannot be equal.

Finally, consider $n \geq 3$. Starting from $5^2 = 4^2 + 3^2$, we get

$$\frac{1}{12^2} = \frac{1}{15^2} + \frac{1}{20^2}$$

by dividing by $3^2 4^2 5^2$. Multiplying by $\frac{1}{12^2}$, we get

$$\frac{1}{12^4} = \frac{1}{12^2 15^2} + \frac{1}{12^2 20^2} = \frac{1}{12^2 15^2} + \left(\frac{1}{15^2} + \frac{1}{20^2}\right)\frac{1}{20^2}$$
$$= \frac{1}{(12 \cdot 15)^2} + \frac{1}{(15 \cdot 20)^2} + \frac{1}{(20 \cdot 20)^2}.$$

Hence

$$(x_1, x_2, x_3, x_4) = (12 \cdot 15, 15 \cdot 20, 20^2, 2 \cdot 12^2)$$

is a solution for $n = 3$. Inductively, assume that $(x_1, \dots, x_{n+1})$ is a solution to

$$\frac{1}{x_1^2} + \dots + \frac{1}{x_n^2} = \frac{n+1}{x_{n+1}^2}$$

for some $n \geq 3$ and arrive in this manner at

$$\frac{1}{x_1^2} + \dots + \frac{1}{x_n^2} + \frac{1}{x_{n+1}^2} = \frac{n+2}{x_{n+1}^2}$$

completing the proof. Remark. For $n = 1$, we get the equation $\sqrt{2}x_1 = x_2$, and since $\sqrt{2}$ is irrational, there is no solution in this case. For $n = 2$, we have

$$x_2^3 x_3^2 + x_1^2 x_3^2 = 3x_1^2 x_2^2$$

or equivalently, $a^2 + b^2 = 3c^2$. We can assume that the numbers a, b and c are all different from zero and that they are relatively prime, meaning $gcd(a, b, c) = 1$. The square of an integer is congruent to 0 or 1 modulo 3, and hence both a and b are divisible by 3. Now, c is also divisible by 3 and we get a contradiction.

For $n = 3$, we have at least one solution:

$$(x_1, x_2, x_3, x_4) = (3,3,6,4),$$

that is

$$\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{6^2} = \frac{1}{4^2}$$

For each integer $n > 3$, we can use the solution for $n = 3$ and get

$$\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{6^2} + \underbrace{\frac{1}{4^2} + \dots + \frac{1}{4^2}}_{n-3} = \frac{4}{4^2} + \frac{n-3}{4^2} = \frac{n+1}{4^2}.$$

Example 5. Prove that for all $n \geq 412$ there are positive integers $x_1, \dots, x_n$ such that

$$\frac{1}{x_1^3} + \frac{1}{x_2^3} + \dots + \frac{1}{x_n^3} = 1. \quad (1)$$

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Solution. We have

$$\frac{1}{a^3} = \frac{1}{(2a)^3} + \cdots + \frac{1}{(2a)^3},$$

where the right-hand side consists of eight summands, so if the equation (1) is solvable in positive integers, then so is the equation

$$\frac{1}{x_1^3} + \frac{1}{x_2^3} + \cdots + \frac{1}{x_{n+7}^3} = 1.$$

Using the method of mathematical induction with pace 7, it suffices to prove the solvability of the equation (1) for $n = 412, 413, \ldots, 418$. The key idea is to construct a solution in each of the above cases from smaller ones modulo 7.

Observe that

$\frac{27}{3^3} = 1$ and $27 \equiv 412 \pmod 7$,

$\frac{4}{2^3} + \frac{9}{3^3} + \frac{36}{6^3} = 1$ and $4 + 9 + 36 = 49 \equiv 413 \pmod 7$,

$\frac{4}{2^3} + \frac{32}{4^3} = 1$ and $4 + 32 = 36 \equiv 414 \pmod 7$,

$\frac{18}{3^3} + \frac{243}{9^3} = 1$ and $18 + 243 = 261 \equiv 415 \pmod 7$,

$\frac{18}{3^3} + \frac{16}{4^3} + \frac{144}{12^3} = 1$ and $18 + 16 + 144 = 178 \equiv 416 \pmod 7$,

$\frac{4}{2^3} + \frac{16}{4^3} + \frac{36}{6^3} + \frac{144}{12^3} = 1$ and $4 + 16 + 36 + 144 = 200 \equiv 417 \pmod 7$.

Finally,

$\frac{4}{2^3} + \frac{9}{3^3} + \frac{81}{9^3} + \frac{324}{18^3} = 1$ and $4 + 9 + 81 + 324 = 418$.

Above,we mentioned the methods of solving problems of different forms. Now I want to mention some problems for independent work in the article.

Problems

1. Prove that for all integers $n \geq 2$ there are odd integers x, y such that $|x^2 - 17y^2| = 4^n$.

(Titu Andreescu)

2. Prove that for all positive integers n, the equation

$$x^2 + xy + y^2 = 7^n$$

is solvable in integers.

(Dorin Andrica)

3. Prove that for each positive integer n, the equation

$$(x^2 + y^2)(u^2 + v^2 + w^2) = 2009^n$$

is solvable in integers.

(Titu Andreescu)

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