TRIGONOMETRIK FUNKSIYALARNI FUNKSIONAL TENGLAMALAR YORDAMIDA ANIQLASH

Shoimov B.S.[1,a], Bozorov M.N[2].
[1]Dotesent, Iqtisodiyot va pedagogika universiteti,
[2] Dotesent, Iqtisodiyot va pedagogika universiteti,
Qarshi shahri, O'zbekiston.
[a]e.mail: boxodir1@bk.ru
[a]https://orcid.org/0009-0009-9697-9973

Annotatsiya: Trigonometrik funksiyalarni geometrik tarzda aniqlashda bu funksiyalarning xossalarini keltirishda geometriyaga tayanishga majburmiz. Ammo maktab matematika kursi o'ta qat'iy talablarga bo'sunmaganligi uchun keltirilgan trigonometriyada qo'llanilgan bir qator isbotlarda qa'tiylik yetishmaydi. Ushbu maqolada trigonometrik funksiyalarni funksional tenglamalar yordamida aniqlash usullari va xossalari o'rganilgan.

Kalit so'zlar: Trigonometrik funksiya, radikal, funksional tenglama, o'suvchi, kamayuvchi, davriy.

ОПРЕДЕЛЕНИЕ ТРИГОНОМЕТРИЧЕСКИХ ФУНКЦИЙ С ПОМОЩЬЮ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ

Аннотация: При геометрическом определении тригонометрических функций нам приходится полагаться на геометрию, чтобы сформулировать свойства этих функций. Однако ряду доказательств в тригонометрии недостает строгости, поскольку школьный курс математики не является строгим. В данной статье изучаются методы и свойства определения тригонометрических функций с помощью функциональных уравнений.

Ключевые слова: Тригонометрическая функция, радикал, функциональное уравнение, возрастающая, убывающая, периодическая.

DETERMINATION OF TRIGONOMETRIC FUNCTIONS USING FUNCTIONAL EQUATIONS

Annotation: In defining trigonometric functions geometrically, we have to rely on geometry to state the properties of these functions. But a number of proofs in trigonometry lack rigor because the school math course is not rigorous. In this article, the methods and properties of determining trigonometric functions using functional equations are studied.

Key words: Trigonometric function, radical, functional equation, increasing, decreasing, periodic.

Geometriya bilan bog'liq bo'lmagan tarzda trigonometrik funksiyalarni aniqlashni differensial tenglamalar asosida amalga oshirish mumkin: a)ikkinchi tartibli chiziqli

$$\frac{d^2y}{dx^2}+y=0$$

tenglamaning

$x=0$ bo'lganda $y_1=1,\ \frac{dy_1}{dx}=0$

boshlang'ich shartlarni qanoatlantiruvchi $y_1(x)$ yechimini kosinus deb nomlaymiz va $\cos x$ kabi belgilaymiz.

b) ikkinchi tartibli chiziqli

$$\frac{d^2y}{dx^2}+y=0$$

tenglamaning

$x=0$ bo'lganda $y_2=0,\ \frac{dy_2}{dx}=1$

boshlang'ich shartlarni qanoatlantiruvchi $y_2(x)$ yechimini sinus deb nomlaymiz va $\sin x$ kabi belgilaymiz.

Trigonometrik funksiyaning keltirilgan bu ta'rifIdan uning barcha xossalarini o'rnatish mumkin va shuningdek x sonini qandaydir burchak o'lchovi sifatida qaralsa $\sin x$ va $\cos x$ funksiyalar yuqorida aytilgan geometrik xossalarga ega bo'ladi.

Trigonometrik funksiyalarni darajali qatorlar yordamida ham aniqlashimiz mumkin.

Bu holda x ning barcha qiymatlarida yaqinlashuvchi

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

darajali qatorning yig'indisiga x soning sinusi, x ning barcha qiymatlarida yaqinlashuvchi

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

darajali qatorning yig'indisiga x soning kosinusi deyiladi. Ammo bu ta'rifni maktab matematika kursiga kiritib bo'lmaydi.

Endi x ning barcha qiymatlarida

$$S(x+y)=S(x)C(y)+C(x)S(y), \qquad (1)$$

$$C(x+y)=C(x)C(y)-S(x)S(y), \qquad (2)$$

$$S^2(x)+C^2(x)=1, \qquad (3)$$

$$\lim_{x\to 0}\frac{S(x)}{x}=1 \qquad (4)$$

shartlarni qanoatlantiruvchi ikkita $S(x)$ va $C(x)$ funksiyalarni qaraymiz. Bu munosabatlardan $S(x)=\sin x$ va $C(x)=\cos x$ funksiyalarni aniqlash mumkin. Biz bu bobda trigonometrik funksiyalarni aniqlashning funksional tenglamalarga asoslangan yana bir usulini keltiramiz. Trigonometrik funksiyalarni aniqlashda π soni muhum o'rin tutadi. Bu sonni analitik ko'rinishda aniqlaymiz.

Ushbu sonlar ketma ketligini qaraymiz:

$$\sqrt{2}, \qquad \sqrt{2+\sqrt{2}}, \qquad \sqrt{2+\sqrt{2+\sqrt{2}}}\ldots$$

Bu ketma ketlikning umumiy hadini z_n bilan belgilaymiz. Bu ketma ketlik monoton o'suvchidir, chunki

$$z_1 < z_2 < z_3 < z_4 < \cdots.$$

Bu ketma ketlikini chegaralangan ekanligini ko'rsatamiz. Haqiqatdan ham, ixtiyoriy natural n soni uchun

$$z_{n+1} = \sqrt{2+z_n}$$

munosabatga ega bo'lamiz. $n=k$ uchun $z_n < 2$ tengsizlik bajarilsin deb faraz qilamiz. U holda $z_{n+1} = \sqrt{2+z_n} < \sqrt{2+2} = 2$. Ammo $z_1 < 2$ bo'lgani uchun $z_2 < 2,\ z_3 < 2$ va hakazo. Bundan z_n sonlar ketma ketligi limitga ega ekanligi kelib chiqadi.

$$\lim_{n\to\infty} z_n = t$$

deb olamiz. Shu bilan birga yana quyidagi

$$2+\sqrt{2}, \qquad 2+\sqrt{2+\sqrt{2}}, \qquad 2+\sqrt{2+\sqrt{2+\sqrt{2}}}\ldots$$

sonlar ketma ketligini qaraymiz. Bu ketma ketlikning umumiy hadi $(z_n)^2$ dan iborat. Shu bilan birga bu ketma ketlikning umumiy hadi $z_{n-1}+2$ ga teng.

Demak,

$$\lim_{n\to\infty}\left[(z_n)^2\right] = \lim_{n\to\infty}(z_n+2)$$

Bu tenglikni $t^2 = t+2$ ko'rinishda yozish mimkin. Bundan $t=2$ bo'lgani uchun

$$\lim_{n\to\infty} z_n = 2$$

natijaga ega bo'lamiz. Endi

$$2\sqrt{2}; \qquad 2^2\sqrt{2-\sqrt{2}}; \qquad 2^3\sqrt{2-\sqrt{2+\sqrt{2}}};\ldots$$

ketma ketlikni olamiz. Bu ketma ketlikning umumiy hadini u_n bilan belgilaymiz.

$$u_n = 2^n\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}$$

(radikallar soni n ta). Bundan

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}\cdot\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}{2^n\cdot\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}$$

(maxrajdagi radikallar soni n ta, suratdagi radikallar soni $n+1$ ta). Bu tenglikning o'ng qismi surat va maxrajini $n+1$ radikallardan iborat $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}$

soniga ko'paytiramiz. Natijada yuqorida $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}<2$ ekanligi ko'rsatilgani uchun radikallar soni nechta bo'lmasin

$$\frac{u_{n+1}}{u_n}=\frac{2^{n+1}\cdot\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}\cdot\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}{2^n\cdot\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}\cdot\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}=$$

$$=\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}>1$$

u_n musbat sonlar bo'lgani uchun $u_{n+1}>u_n$. Demak, u_n ketma ketlik monoton o'suvchidir.

Shu bilan birga quyigagi sonlar ketma ketligini qaraymiz:

$$\frac{2^2\sqrt{2}}{\sqrt{2}};\quad \frac{2^3\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}};\ \frac{2^4\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}};\ \frac{2^5\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}};\ldots.$$

Bu ketma ketlikning umumiy hadini v_n orqali belgilaymiz:

$$v_n=\frac{2^n\cdot\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}$$

(maxrajdagi radikallar soni ham n ta, suratdagi radikallar soni ham n ta).

Xuddi yuqorida $\frac{u_{n+1}}{u_n}$ nisbatga nisbatan bajarilgan ishlarga o'xshash tarzda

$$\frac{v_n}{v_{n+1}}=\frac{1}{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}+\frac{1}{2}$$

ekanligini ko'rsatamiz.

$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}<2$ bo'lgani uchun $\frac{v_n}{v_{n+1}}>1$ bo'ladi. v_n musbat sonlar bo'lgani ucgun $v_{n+1}<v_n$ bo'lib, v_n sonlar ketma ketligi kamayuvchi bo'ladi. Endi ixtiyoriy natural n soni uchun $u_n<v_n$ ekanligini ko'rsatamiz. Haqiqatdan ham,

$$\frac{u_n}{v_n}=\frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{2}$$ (suratdagi radikallar soni n ta).

$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}<2$ bo'lgani uchun $\frac{u_n}{v_n}<1$. Demak n ortishi bilan u_n ketma ketlik o'suvchi bo'lada, ammo qandaydir sondan masalan $v_1 = \frac{2^2\sqrt{2}}{\sqrt{2}} = 4$ sonidan kichikligicha qoladi, chunki $u_n < v_n < v_2 = 4$. Bundan u_n ketma ketlik limiotga ega. Biz bu limitni π bilan belgilaymiz.

Demak,

$$\pi = \lim_{n\to\infty} 2^n \cdot \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$$ (radikallar soni n ta).

Ikkinchi tomondan n ortishi bilan v_n ketma ketlik kamayuvchi bo'lada, ammo qandaydir sondan masalan $u_1 = 2\sqrt{2}$ sonidan kattaligicha qoladi, chunki $v_n > u_n > u_1 = 2\sqrt{2}$. Demak, v_n ketma ketlik limitga ega. Shu bilan birga

$$\frac{v_n}{u_n} = \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}$$

bo'lgani uchun

$$\lim_{n\to\infty}\frac{v_n}{u_n} = 1 \quad \text{va} \quad \lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n = \pi$$

Geometrik nuqtai nazardan u_n va v_n ketma ketlikliklar mos ravishda $R = 1$ radiusli aylanaga ichki va tashqi chizilgan muntazam n burchaklarning yarim perimetrlaridir. Shuning uchun π aylana uzunligini diametrga nisbatidir.

Funksional tenglamalarning dastlabki sistemasi.

Funksional tenglamalarning

$$S(x-y) = S(x)\cdot C(y) - C(x)\cdot S(y), \qquad (5)$$

$$C(x-y) = C(x)\cdot C(y) + S(x)\cdot S(y), \qquad (6)$$

sistemalarini

$$x \in \left(0, \frac{\pi}{2}\right) \qquad (7)$$

oralikda

$$S\left(\frac{\pi}{2}\right) = 1 \qquad (8)$$

shart bo'yicha qaramiz va bu shartlarni qanoatlantiruvchi $S(x)$ va $C(x)$ funksiysyalar mavjud deb faraz qilamiz. (5)-(8) munosabatlardan $S(x)$ va $C(x)$ funksiylarning quyidagi xossalari kelib chiqadi.

(4) munosabatda $x = y$ deb olib

$$S(0) = 0 \qquad (9)$$

munosabatga ega bo'lamiz. (4) munosabatda $x = \frac{\pi}{2}$, $y = 0$ deb olib. (9) munosabatni e'tiborga olgan holda

$$C(0) = 1$$

munosabatga ega bo'lamiz. (5) tenglikda $x = y$ deb olib,

$$S^2(x) + C^2(x) = 1 \qquad (10)$$

tenglikni hosil qilamiz. (10) munosabatdan

$$|S(x)| \le 1, \quad |C(x)| \le 1 \qquad (11)$$

hamda (10) dan $x = \frac{\pi}{2}$ bo'lganda

$$S^2\left(\frac{\pi}{2}\right) + C^2\left(\frac{\pi}{2}\right) = 1$$

tenglikni hosil qilamiz. Ammo $S\left(\frac{\pi}{2}\right) = 1$ bo'lgani uchun

$$C\left(\frac{\pi}{2}\right) = 0 \qquad (12)$$

munosabat o'rinli bo'ladi.

$S(x)$ funksiyani toq funksiya, $C(x)$ funksiyani juft funksiya ekanligini ko'rsatamiz. Hqiqatdan ham, (5) munosabatda $x = 0$ deb olib,

$$S(-y) = S(0) \cdot C(y) - C(0) \cdot S(y)$$

va bundan $S(0) = 0,\ C(0) = 1$ bo'lgani uchun

$$S(-y) = -S(y) \qquad (13)$$

munosabat o'rinli bo'ladi. (6) tenglikda $x = 0$ deb olsak,

$$C(-y) = C(0) \cdot C(y) + S(0) \cdot S(y)$$

va bundan $S(0) = 0,\ C(0) = 1$ bo'lgani uchun

$$C(-y) = C(y) \qquad (14)$$

munosabat o'rinli bo'ladi.

Endi $S(x)$ va $C(x)$ funksiyalar uchun qo'shish formulalarini keltirib chiqarish qiyin emas. Haqiqatdan ham, (5) va (6) formulalarda y ni $-y$ bilan almashtirib,

$$S(x+y) = S(x) \cdot C(-y) - C(x) \cdot S(-y), \qquad (15)$$

$$C(x+y) = C(x) \cdot C(-y) + S(x) \cdot S(-y), \qquad (16)$$

tengliklarni hosil qilamiz. (15) va (16) qo'shish formulalarida $y = \frac{\pi}{2}$ bo'lganda quyidagi formulalar kelib chiqadi:

$$S\left(x + \frac{\pi}{2}\right) = C(x) \qquad (17),$$

$$C\left(x + \frac{\pi}{2}\right) = -S(x) \qquad (18)$$

Bu ikki formula (13) va (14) formulalar bilan birgalikda x ning ixtiyoriy haqiqiy qiymati uchun $S(x)$ va $C(x)$ funksiyalarning qiymatlarini bu funksiyalarning x ning $\left(0, \frac{\pi}{2}\right)$ oraliqdagi qiymatlari orqali topish imkoniyatini beradi. Xususiy holda

$$S(\pi+x)=C\left(x+\frac{\pi}{2}\right)=-S(x),\qquad C(\pi+x)=-S\left(x+\frac{\pi}{2}\right)=-C(x)$$

$S(x)$ va $C(x)$ funksiyalarning davriyligi va ularning asosiy xossalari

Endi yuqoridagilardan foydalanib, $S(x)$ va $C(x)$ funksiyalarning davriyligini ko'rsatishimiz mumkin. Haqiqatda quyidagi teorema o'rinli:

1-teorema. $S(x)$ funksiya davriy va 2π soni bu funksiyaning eng kichik musbat davri bo'ladi.

Isboti. $S(2\pi+x)=S[\pi+(\pi+x)]=-S(\pi+x)=S(x)$ bo'lgani uchun har qanday x soni uchun

$$S(2\pi+x)=S(x)$$

tenglikka ega bo'lamiz.

Faraz qilaylik shunday k $(0<k<2\pi)$ soni mavjudki, har qanday x soni uchun

$$S(k+x)=S(x) \qquad (*)$$

tenglik o'rinli bo'lsin. k soni π soniga teng emas, chunki agar $k=\pi$ bo'lsa, u holda

$$S(\pi+x)=S(x) \qquad (*)$$

Ammo $S(\pi+x)=-S(x)$ bo'lgani uchun $S(x)=-S(x)$ bo'lib, bundan (3) ga zid bo'lgan $S(x)=0$ ziddiyatni hosil qilamiz.

k soni $\frac{3\pi}{2}$ soniga ham teng emas, chunki agar $k=\frac{3}{2}\pi$ bo'lsa, u holda

$$S\left(\frac{3}{2}\pi+x\right)=S(x)$$

Ammo

$$S\left(\frac{3}{2}\pi+x\right)=S\left[\frac{\pi}{2}+(\pi+x)\right]-C(\pi+x)=-C(-x)=-C(x)$$

bo'lgani uchun biz $S(x)=-C(x)$ ziddiyatga ega bo'lamiz.

k soni $\frac{\pi}{2}$ soniga ham teng emas, chunki agar $k=\frac{\pi}{2}$ bo'lsa, u holda

$$S\left(\frac{\pi}{2}+x\right)=S(x)$$

Ammo boshqa tomondan $S\left(x+\frac{\pi}{2}\right)=C(x)$ bo'lgani uchun biz $S(x)=C(x)$ ziddiyatga ega bo'lamiz. $(*)$ munosabatda $x=0$ deb olib $S(k)=S(0)$ tenglikni hosil qilamiz.

k soni $\left(0,\frac{\pi}{2}\right)$ oraliqqa ham tegishli emas, aks holda biz (3) ga zid munosabatga kelamiz.

k soni $\left(\frac{\pi}{2},\pi\right)$ oraliqqa ham tegishli emas, chunki $\pi-k$ soni $\left(0,\frac{\pi}{2}\right)$ oraliqqa tegishli bo'lib, biz (3) ga zid bo'lgan $S(\pi-k)=-S(-k)=S(k)=0$ munosabatga kelamiz.

k soni $(\pi,2\pi)$ oraliqqa ham tegishli emas, chunki $2\pi-k$ soni $(0,\pi)$ oraliqqa tegishli bo'lib, biz (3) ga zid bo'lgan

$$S(2\pi-k)=S[2\pi+(-k)]=S(-k)=-S(k)=0$$

munosabatga kelamiz.

Demak, 2π soni $S(x)$ funksiyaning eng kichik musbat davri ekan.

2-**teorema.** $C(x)$ funksiya davriy va 2π soni bu funksiyaning eng kichik musbat davri bo'ladi.

Isboti. $C(2\pi+x)=C[\pi+(\pi+x)]=-C(\pi+x)=C(x)$ bo'lgani uchun har qanday x soni uchun $C(2\pi+x)=C(x)$ tenglikka ega bo'lamiz.

Faraz qilaylik shunday k $(0<k<2\pi)$ soni mavjudki, har qanday x soni uchun

$$C(k+x)=C(x) \qquad (**)$$

tenglik o'rinli bo'lsin. k soni $\frac{1}{2}\pi$ soniga teng emas, chunki agar $k=\frac{1}{2}\pi$ bo'lsa, u holda

$$C\left(\frac{1}{2}\pi+x\right)=C(x) \qquad (*)$$

Ammo $C\left(\frac{1}{2}\pi+x\right)=-S(x)$ bo'lgani uchun $C(x)=-S(x)$ ziddiyatni hosil qilamiz.

k soni π soniga ham teng emas, chunki agar $k=\pi$ bo'lsa, u holda $C(\pi+x)=C(x)$.

Ammo $C(\pi+x)=-C(x)$ bo'lgani uchun biz $C(x)=-C(x)$ tenglikdan $C(x)=0$ ziddiyatga ega bo'lamiz.

k soni $\frac{3\pi}{2}$ soniga ham teng emas, chunki agar $k=\frac{3\pi}{2}$ bo'lsa, u holda

$$C\left(\frac{3\pi}{2}+x\right)=C(x)$$

Ammo boshqa tomondan $C\left(x+\frac{3\pi}{2}\right)=C\left[\frac{\pi}{2}+(\pi+x)\right]=-S(\pi+x)=S(x)$ bo'lgani uchun biz $C(x)=S(x)$ ziddiyatga ega bo'lamiz, chinki $C(0)=1,\ S(0)=0$. Demak, $k\neq\frac{\pi}{2},\ k\neq\pi,\ k\neq\frac{3\pi}{2}$. $(**)$ munosabatda $x=\frac{\pi}{2}$ deb olinsa

$$C\left(\frac{\pi}{2}+k\right)=C\left(\frac{\pi}{2}\right)=0$$

tenglikni hosil qilamiz. Ammo $C\left(x+\frac{\pi}{2}\right)=-S(k)$. Demak, $S(k)=0$. Lekin bu munosabatning bo'lishi mumkin emasligi yuqorida ko'rsatildi.

Demak, 2π soni $C(x)$ funksiyaning eng kichik musbat davri ekan.

Biz kelgusida (1) va (2) qo'shish formulalaridan kelib chiquvchi quyidagi

$$S(x_1)+S(x_2)=2S\left(\frac{x_1+x_2}{2}\right)\cdot C\left(\frac{x_1-x_2}{2}\right),$$

$$S(x_1)-S(x_2)=2C\left(\frac{x_1+x_2}{2}\right)\cdot S\left(\frac{x_1-x_2}{2}\right)$$

$$C(x_1)+C(x_2)=2C\left(\frac{x_1+x_2}{2}\right)\cdot C\left(\frac{x_1-x_2}{2}\right),$$

$$C(x_1)-C(x_2)=2S\left(\frac{x_1+x_2}{2}\right)\cdot S\left(\frac{x_1-x_2}{2}\right)$$

formulalarni keltirib chiqaramiz.

Masalan, birinchi formula quyidagi tartibda hosil qilinadi.

$$S(x_1)+S(x_2)=S\left(\frac{x_1+x_2}{2}+\frac{x_1-x_2}{2}\right)+S\left(\frac{x_1+x_2}{2}-\frac{x_1-x_2}{2}\right)=2S\left(\frac{x_1+x_2}{2}\right)\cdot C\left(\frac{x_1-x_2}{2}\right)$$

Keyin,

$$S(2x)=S(x)\cdot C(x)+C(x)\cdot S(x)=2S(x)\cdot C(x), \qquad C(2x)=C^2(x)-S^2(x)$$

formulalarga ega bo'lamiz. Nihoyat

$$S\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1-C(x)}{2}}, \qquad C\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1+C(x)}{2}}$$

formulalarni keltirib chiqaramiz.

Haqiqatdan ham,

$$1-C(x)=2S\left(\frac{x}{2}\right)\cdot S\left(\frac{x}{2}\right)=2S^2\left(\frac{x}{2}\right)$$

bo'lgani uchun

$$S\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1-C(x)}{2}}$$

Keyin

$$1+C(x)=2C\left(\frac{x}{2}\right)\cdot C\left(-\frac{x}{2}\right)=2C^2\left(\frac{x}{2}\right)$$

bo'lgani uchun

$$C\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1+C(x)}{2}}$$

3-teorema. $C(x)$ funksiya $\left(0;\frac{\pi}{2}\right)$ oraliqda musbat.

Isboti. Agar $0<x<\frac{\pi}{2}$ bo'lsa, u holda $0<\frac{\pi}{2}-x<\frac{\pi}{2}$. Faraz qilaylik $C(x)\le 0$ bo'lsin. U holda $S\left(\frac{\pi}{2}-x\right)\le 0$ bo'lib, bu (3) ga zid. Demak, $0<x<\frac{\pi}{2}$ bo'lganda $C(x)>0$ ekan.

4-teorema. $\left[0,\frac{\pi}{2}\right]$ kesmada $S(x)$ funksiya o'suvchi, $C(x)$ funksiya esa kamayuvchi funksiya bo'ladi.

Isboti. Aytaylik, x_1 va x_2 sonlari $\left[0,\frac{\pi}{2}\right]$ kesmaga tegishli bo'lib, $x_1 > x_2$ bo'lsin. U holda

$$0<\frac{x_1+x_2}{2}<\frac{\pi}{2} \quad \text{va} \quad 0<\frac{x_1-x_2}{2}<\frac{\pi}{2}$$

Demak,

$$S\left(\frac{x_1-x_2}{2}\right)>0 \quad \text{va} \quad C\left(\frac{x_1+x_2}{2}\right)>0$$

Shu bilan birga

$$S(x_1)-S(x_2)=2C\left(\frac{x_1+x_2}{2}\right)\cdot S\left(\frac{x_1-x_2}{2}\right)$$

bo'lgani uchun $S(x_1)-S(x_2)>0$, ya'ni $S(x_1)>S(x_2)$bo'lib $S(x)$ funksiya o'suvchi bo'ladi.

Keyin

$$C(x_1)-C(x_2)=2S\left(\frac{x_1+x_2}{2}\right)\cdot S\left(\frac{x_2-x_1}{2}\right)=-2S\left(\frac{x_1+x_2}{2}\right)\cdot S\left(\frac{x_1-x_2}{2}\right)<0$$

bo'lgani uchun $C(x_1)-C(x_2)<0$, ya'ni $C(x_1)<C(x_2)$bo'lib $C(x)$ funksiya kamayuvchi bo'ladi.

5-teorema. Ixtiyoriy n natural soni uchun

$$2C\left(\frac{\pi}{2^n}\right)=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \qquad (*)$$

$$2S\left(\frac{\pi}{2^n}\right)=\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \qquad (**)$$

tengliklar o'rinli, bu yerda radikallar soni $n-1$ ta.

Isboti. Ma'lumki,

$$2C\left(\frac{\pi}{4}\right)=2\cdot\sqrt{\frac{1+C\left(\frac{\pi}{2}\right)}{2}}=\sqrt{2};\quad 2C\left(\frac{\pi}{8}\right)=2\cdot\sqrt{\frac{1+C\left(\frac{\pi}{4}\right)}{2}}=2\cdot\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\sqrt{2+\sqrt{2}}$$

Aytaylik $(*)$ munosabat $n=k$ uchun to'g'ri bo'lsin. U holda $k-1$ ta radikallar uchun

$$2C\left(\frac{\pi}{2^k}\right)=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$$

bo'lishi kerak. Natijada

$$2C\left(\frac{\pi}{2^{k+1}}\right)=2\sqrt{\frac{1+C\left(\frac{\pi}{2^k}\right)}{2}}=2\sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{2}}{2}}=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$$

bo'lib, bu yerda radikallar soni k ta. Demak, $(*)$ munosabat $n=k+1$ uchun ham to'g'ri. Bu tenglik $n=2$ uchun ham to'g'ri bo'lgani uchun ixtiyoriy natural n uchun ham o'rinli. Ikkinchi tenglik esa quyidagi mulohazalardan kelib chiqadi:

$$2S\left(\frac{\pi}{2^n}\right)=2\sqrt{1-C^2\left(\frac{\pi}{2^n}\right)}=2\sqrt{1-\frac{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{4}}=\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$$

radikallar soni $n-1$ tadan iborat.

6-teorema. Agar n natural son bo'lsa, u holda

$$\lim_{n\to\infty}C\left(\frac{\pi}{2^n}\right)=1 \quad \text{va} \quad \lim_{n\to\infty}S\left(\frac{\pi}{2^n}\right)=0$$

Isboti. Haqiqatdan ham, biz yuqorida, radikallar soni n ta bo'lganda

$$z_n=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$$

belgilash kiritgan edik. Shung uchun $C\left(\frac{\pi}{2^n}\right)=\frac{z_n-1}{2}$

Bundan

$$\lim_{n\to\infty}C\left(\frac{\pi}{2^n}\right)=\lim_{n\to\infty}\frac{z_n-1}{2}=1,$$

chunki $\lim_{n\to\infty}z_{n-1}=2$

Ikkinchi tomondan yuqorida belgilashlarni e'tiborga olinsa

$$S\left(\frac{\pi}{2^n}\right)=\frac{\sqrt{2-z_{n-2}}}{2}$$

Bundan esa talab etilayotgan ikkinchi munosabat kelib chiqadi.

7-teorema. $C(x)$ va $S(x)$ funksiyalar uzluksizdir.

Isboti. $\Delta C(x)=C(x+\Delta x)-C(x)=-2S\left(\frac{\Delta x}{2}\right)\cdot S\left(x+\frac{\Delta x}{2}\right)$ va shu bilan birga

$\left|S\left(x+\frac{\Delta x}{2}\right)\right|\le 1$ va $\lim_{\Delta x\to 0}S\left(\frac{\Delta x}{2}\right)=0$ bo'lgani uchun $\lim_{\Delta x\to 0}\Delta C(x)=0$.

$\Delta S(x)=S(x+\Delta x)-S(x)=2S\left(\frac{\Delta x}{2}\right)\cdot C\left(x+\frac{\Delta x}{2}\right)$ va shu bilan birga

$\left|C\left(x+\frac{\Delta x}{2}\right)\right|\le 1$ va $\lim_{\Delta x\to 0}S\left(\frac{\Delta x}{2}\right)=0$ bo'lgani uchun $\lim_{\Delta x\to 0}\Delta S(x)=0$

FOYDALANILGAN ADABIYOTLAR:

1. Maxmudova D, Do'stmurodova G "Qiziqarli matematika va olimpiada masalalari"Chirchiq 2000 y

2.B.A.Shoimqulov, R.M.Madrahimov, N.B.Kamolov "Talabalarning matematikadan olimpiada masalalari" Toshkent – 2013.

3.Ismoilov U "Matematikadan olimpiada masalalari" Toshkent "Yangi avlod"

nashriyoti 2007-y
4. Artur Engel Problem-Solving Strategies. 1998 Springer-Verlag New York
5. Лопщиц А.М. Функциональные уравнения.-Квант,1970 г. №1-2,30-35 с.
6. Котельников П.М. О функциональных уравнениях, определяющих тригонометрические функции. Математика в школе , 1951, №2, 1-12 с.
7. Ясинский В.А. Олимпиадна математика, функциональни ривнення.Х.: Основа, 2005.
8. Андреев А.А. и др. Функциональные уравнения.-Самара: В мире науки,1999.
9. Paul Vaderlind. Functional Equations for The Beginners. Stockholm University, 2005.
10. Titu Andreecu, Iurie Borieico. Functional Equations. Electronic Edition, 2007.