COMPOSITION FUNCTIONS MEASURABLE

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In this paper, the stability of a class of nonlinear integro-differential equation is investigated and analyzed. By defining a suitable Lyapunov functional we establish necessary and sufficient condition -for the stability of the zero solution. Our results extends known results in the literature.

Keywords: Integro-differential equation, Lyapunov functional, Nonlinear, Stability.

1. Introduction

Volterra integro-differential equations have wide applications in biology, ecology, medicine, physics and other scientific areas and thus has been extensively studied. The equilibrium or the steady state of a linear or nonlinear equation can either be stable or unstable. The steady state is called a stable system if after been disturbed by some physical phenomenon returns to its uniform state of rest or its normal position. When a system tends to a new position after a slight displacement, such equilibrium is called unstable equilibrium.

The origin of stability in science and engineering can be track down to the work of Aristotle and Archimedes (Magnus 1959). Alexander Lyapunov was the first to define the notion of stable system in Mathematical form in 1892, in his book on "the general problem of stability".

The stability theorem for motion studied by A.M Lyapunov has proven to be highly useful and applicable in the field of science and engineering. The notion of stability is studied in the literature under three classes, namely; Bounded input and bounded output (BIBO), Zero-input stability and Input-state Stability. Over the years, Lyapunov method for the stability of integro-differential equation have been proposed by different researcher (Stamove and Stomov (2001, 2013), Tunc (2016), Tunc and Sizar (2017) Vanulailai and Nakagiri (2003), Carabollo et al. (2007),

Here we are interested in sufficient conditions on $f:(\text{Ў},a)$ ® $(\text{Ў},b)$ that can guarantee that if $g:(\text{Ў},a)$ ® $(\text{Ў},b)$ is measurable, so is $g \circ f:(\text{Ў},a)$ ® $(\text{Ў},a)$. We will consider then m^* and m, the Lebesgue exterior measure and measure respectively. For brevity, we will speak of a/b and a/a measurable functions.

Rather counterintuitively, great regularity or monotonicity of f does not guarantee that the composition $g \circ f$ will be a/b measurable when g is. We illustrate this point with the following example.

Example 1.2. We will first proceed to construct a strictly increasing C^{∞} function f which is not a/a measurable, that is, such that there exists $D \in a$ such that $f^{-1}(D) \notin a$.

Let $C \subset I := [0,1]$ be a Smith-Volterra-Cantor set such that $m(C) \in (0,1)$ [3, p.39] and consider the function

$$y(x) := e^{-(1-x^2)^{-1}}$$

for $x \in (-1,1)$. $y \in C^{\infty}((-1,1),\mathbb{R})$. Since C is closed, $I \setminus C$ is open, so it has a sountable number of connected components each of which is an open interval. Let us define $h(x) = 0$ if $x \in C$ and

$$h(x) = 2^{-(b-a)^{-1}} y\left(\frac{2x-b-a}{b-a}\right)$$

if $x \in (a,b)$ where (a,b) is a connected component of $I \setminus C$. Let $M_n := \max\left|y^{(n)}\right|$. We will show now that $h \in C^{\infty}(I,\mathbb{R})$. It is clear that h is C^{∞} in $I \setminus C$. If $x \in C$, let us check that $\lim_{y \to x^-} f(x) = 0$ (in case the limit can be taken). If $x = b$ for some $(a,b) \subset I \setminus C$, this is obvious. Otherwise, there is a sequence of points in C converging to x from the left. Thus, given $e \in \mathbb{R}^+$, there exists $y \in C$, $x - e < y < x$, so any $z \in (y,x) \setminus C$ belongs to an interval (a,b) with $b - a < e$ and, therefore,

$$h(z) = 2^{-(b-a)^{-1}} y\left((2x-b-a)/(b-a)\right) < 2^{-(b-a)^{-1}} < b-a < e.$$

Hence, $\lim_{y \to x^-} f(x) = 0$. Repeating the argument for limits from the right and observing that $h \setminus c = 0$, we conclude that h is continuous. Assuming h is $n-1$ times differentiable and taking into account that

$$\left|h^{(n)}(x)\right| = 2^{-(b-a)^{-1}} 2^n (b-a)^{-n} \left|y^{(n)}\left(\frac{2x-b-a}{b-a}\right)\right| \le 2^{-(b-a)^{-1}} 2^n (b-a)^{-n} M_n,$$

for any $x \in (a,b) \subset I \setminus C$, we can reason as before to conclude that $h \in C^{\infty}(I,\mathbb{R})$.

Define $f(x) = \int_0^y h(x)\,dy$. $f \in C^{\infty}(I,\mathbb{R})$ and f is strictly increasing. Indeed, since C is totally disconnected, given $x,y \in I$, $x < y$, there exist $t,s \in (x,y)$, $t < s$ such that $[t,s] \subset I \setminus C$, so $h(x) > 0$ in $[t,s]$ and hence $f(y) - f(x) = \int_x^y h(z)dz > 0$.

Given a connected component (a,b) of $I \setminus C$,

$$m(f((a,b))) = f(b) - f(a) = \int_a^b h(z)dz.$$

Thus, given that f is strictly increasing, $I \setminus C$ has a countable number of connected components and that the Lebesgue measure is σ - additive, $m(f(I \setminus C)) = \int_{I \setminus C} h(z)dz$. $f(I)$ is also measurable because it is an interval. Since f is strictly increasing, $f(C) \cap f(I \setminus C) = \emptyset$ and $f(C) = f(I) \setminus f(I \setminus C)$ and, therefore, $f(C)$ is measurable. Hence,

$$m(f(C)) = m(f(I \setminus C)) = f(1) - f(0) - \int_{I \setminus C} h(z)dz = \int_0^1 h(z)dz - \int_{I \setminus C} h(z)dz = \int_C h(z$$

Lemma 1.3. *If g is α / β measurable, f invertible and f^{-1} is absolutely continuous, then $g \circ f$ is α / β measurable.*

Proof. Let $B \in \beta$. Then $g^{-1}(B) \in \alpha$. Since f^{-1} is absolutely continuous it takes α sets to α sets [7 p. 250], so $f^{-1}(g^{-1}(B)) \in \alpha$.

Observe that Lemma 1.3 crucially avoids the circumstances of Example 1.2, since, in that case, the function f^{-1} was not absolutely continuous (since it did not map α sets to α sets). This illustrates that, in general, even if f is absolutely continuous, f^{-1} needs not to be –see [1,9]. A necessary sufficient sondition for f^{-1} to be absolutely continuous (in the case the domain is an interval) can be found in the following result –cf. [2, Lemma 2.2], [9].

$$\sum_{n=m+1}^{\infty} (b_n - a_n) < \delta.$$

Let $X = \bigcup_{n=1}^{m} [a_n, b_n)$. Then, $m^*(E_k \setminus X) < \delta$ since $\{[a_n, b_n)\}_{n=m+1}^{\infty}$ is a cover of $E_k \setminus X$. Thus, Take $x \in U \setminus V$. Then there exist $r \in \mathbb{R}^{+}$ such that $(x-r, x+r) \subset U$. Since $x \in C, x \in \partial(I \setminus C)$, so there exists $y \in (x-r, x+r) \setminus C$. Since $I \setminus C$ is open, there exists $s \in \mathbb{R}^{+}$ such that $(y-s, y+s) \subset (x-r, x+r) \setminus C$. Thus, $m(3(x-r, x+r)) = m(((x-r, x+r) \setminus V) \cup W) = 2r$. Therefore, $m((x-r, x+r) \setminus C) = 0$ and, since $(y-s, y+s) \subset (x-r, x+r) \setminus C$, we have that $m((y-s, y+s)) = 0$, which is a contradiction.

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